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Curvature Diagnostics in Lorentzian Geometry with Applications to General Relativity

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ABSTRACT: Scalar curvature invariants are coordinate independent, but their ability to detect curvature and identify geometric structure depends on the class of Lorentzian metrics under consideration. This focused survey compares scalar and observer-dependent tidal diagnostics in four-dimensional general relativity. After fixing the curvature convention, we give explicit proofs for three model classes. For Schwarzschild space-time, the Kretschmann scalar and the squared norm of its gradient recover the areal radius and positive mass parameter within the specified solution family. The gradient invariant vanishes at the horizon, while a regular coordinate extension establishes that the horizon is not a curvature singularity. For homogeneous and isotropic metrics, the Kretschmann scalar separates temporal and spatial curvature contributions and provides a class-specific flatness criterion. An exact vacuum plane wave supplies a counterexample to any unrestricted inference from vanishing scalar polynomial curvature invariants to flatness: its algebraic curvature contractions vanish while its tidal operator remains nonzero. The examples are related to established work on invariant characterization, stationary-horizon diagnostics and low-regularity singularity theorems. They show why a diagnostic must be accompanied by metric-class assumptions, an observer specification where appropriate, and a distinction between local curvature and global causal structure. The contribution is a comparative mathematical exposition with explicit derivations; no new classification theorem or observational result is claimed.

KEYWORDS: Lorentzian geometry, curvature invariants, geodesic deviation, Schwarzschild space-time, plane waves, general relativity.

1. INTRODUCTION

The relation between differential geometry and general relativity is expressed through geometric quantities that perform different mathematical tasks. A Lorentzian metric specifies intervals and causal directions. Its Levi-Civita connection defines covariant differentiation and geodesic motion. The Riemann tensor measures the failure of covariant derivatives to commute and determines relative acceleration between nearby freely falling trajectories. Curvature contractions enter the Einstein equations, while scalar invariants can diagnose properties of particular solutions (Carroll, 2004; Tong, 2019). These uses should not be treated as interchangeable. Connection coefficients are coordinate dependent. A vanishing Ricci tensor is compatible with nonzero Riemann curvature. A scalar obtained by contracting a Lorentzian tensor is not generally a positive-definite norm. Consequently, familiar diagnostic statements that hold in a symmetric example can fail for another metric class. An explanation of general relativity that moves directly from a scalar calculation to a physical conclusion can therefore conceal the assumptions doing the essential work. Invariant characterization is an established research subject. Coley, Hervik and Pelavas (2009) distinguish Lorentzian metrics that are locally characterized by their scalar polynomial curvature invariants from exceptional Kundt behaviour. Stationary-horizon diagnostics introduce derivatives of invariants, rather than relying on a single algebraic contraction (Page and Shoom, 2015). Recent low-regularity developments address a separate but related issue: how much differentiability is required to formulate curvature, geodesics and incompleteness results (Kunzinger, Ohanyan, Schinnerl and Steinbauer, 2022; Steinbauer, 2023). The present survey does not claim to resolve those classification or regularity problems. Its narrower purpose is to compare three diagnostically different classes using a consistent set of definitions and proofs. Schwarzschild space-time shows how two invariants recover geometric parameters under a specified family assumption. Homogeneous cosmology supplies a positive sum-of-squares flatness criterion within another restricted class. An exact plane wave shows why that criterion cannot be exported to arbitrary Lorentzian metrics. Together these examples turn a general account of geometry in relativity into a precise question: what does a chosen curvature diagnostic establish, under which hypotheses, and what information does it omit?

1.1 Scope and contribution

The article is a focused mathematical survey with worked derivations. The propositions below are standard identities or immediate consequences of established solutions, organized to support a comparative argument; their numbering does not assert historical priority. The main expository contribution is the explicit separation of algebraic curvature detection, family-dependent invariant reconstruction, observer-relative tidal effects and global singularity claims. No universal flatness test, new exact solution or new singularity theorem is proposed. The analysis uses smooth four-dimensional metrics, except when discussing cited low-regularity results. It is not a systematic literature review and contains no observational analysis or numerical evolution. These restrictions allow the hypotheses and conclusions of each calculation to remain inspectable.

2 GEOMETRIC CONVENTIONS AND DIAGNOSTIC OBJECTS

2.1 Curvature and field equations

Let (M, g) be a smooth time-oriented four-dimensional Lorentzian manifold with signature $((-+++))$. Greek indices range from 0 to 3; Latin spatial indices range from 1 to 3. Repeated indices are summed. Units have $(c=G=1)$ except where restored explicitly. The Levi-Civita connection is defined by metric compatibility and vanishing torsion, with coefficients

$$\Gamma_{\mu\nu}^{\rho} = \frac{1}{2} g^{\rho\sigma} (\partial_{\mu} g_{\sigma\nu} + \partial_{\nu} g_{\sigma\mu} - \partial_{\sigma} g_{\mu\nu}). \quad (1)$$

Our curvature convention is

$$R_{\sigma\mu\nu}^{\rho} = \partial_{\mu} \Gamma_{\nu\sigma}^{\rho} - \partial_{\nu} \Gamma_{\mu\sigma}^{\rho} + \Gamma_{\mu\lambda}^{\rho} \Gamma_{\nu\sigma}^{\lambda} - \Gamma_{\nu\lambda}^{\rho} \Gamma_{\mu\sigma}^{\lambda}. \quad (2)$$

Thus $[\nabla_{\mu}, \nabla_{\nu}]V^{\rho} = R_{\sigma\mu\nu}^{\rho} V^{\sigma}$. Set $R_{\sigma\nu} = R_{\sigma\rho\nu}^{\rho}$ and $R = g^{\sigma\nu} R_{\sigma\nu}$. The Einstein equation is

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \quad (3)$$

For zero cosmological constant, vacuum means $R_{\mu\nu} = 0$, not $R_{\alpha\beta\gamma\delta} = 0$. The Weyl tensor then equals the Riemann tensor. Equation (3) supplies dynamics; coordinate invariance alone does not select those dynamics or determine a matter model.

2.2 Scalar and tidal diagnostics

A scalar polynomial curvature invariant is a complete contraction of a polynomial in the Riemann tensor and, when allowed, its covariant derivatives. An algebraic invariant uses no covariant derivatives. Two diagnostics used below are

$$\mathcal{K} = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}, \quad \mathcal{D} = g^{\mu\nu} (\nabla_{\mu} \mathcal{K}) (\nabla_{\nu} \mathcal{K}). \quad (4)$$

The first is the Kretschmann scalar. The second is a differential invariant and can have either sign because the metric is Lorentzian. Neither sign nor magnitude should be interpreted as a Euclidean norm without additional hypotheses.

Let U be a unit timelike vector at an event and let e_i be an orthonormal basis of its spatial orthogonal complement. Define the tidal matrix

$$E_{ij} = R_{\alpha\mu\beta\nu} e_i^{\alpha} U^{\mu} e_j^{\beta} U^{\nu}. \quad (5)$$

For an infinitesimal separation field along an affinity parametrized geodesic congruence, geodesic deviation is

$$\frac{D^2 \xi^{\rho}}{d\tau^2} = -R_{\sigma\mu\nu}^{\rho} U^{\sigma} \xi^{\mu} U^{\nu}. \quad (6)$$

In a parallel-propagated orthonormal frame, the spatial relative acceleration is $\ddot{\xi}^i = -E_j^i \xi^j$. The matrix depends on the observer's time like direction; its eigenvalues are unchanged by rotating that observer's spatial basis. This makes it an observer-defined tensor diagnostic rather than a scalar invariant of the metric alone.

2.3 What each diagnostic can establish

Flatness means vanishing Riemann curvature on the region considered. If the metric is smooth and curvature vanishes there, each sufficiently small neighborhood is isometric to a region of Minkowski space-time. This local statement does not establish a global Minkowski topology or global completeness. Conversely, nonzero Riemann components in a well-defined frame establish local nonflatness even if particular scalar contractions vanish. The distinction becomes explicit in Section 5.3 Schwarzschild space-time and invariant reconstruction

3.1 Curvature and tidal eigenvalues

Consider the positive-mass Schwarzschild family with zero cosmological constant:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2, \quad f(r) = 1 - \frac{2m}{r}, \quad m > 0. \quad (7)$$

The coordinate r is the areal radius, so symmetry spheres have area $4\pi r^2$. The parameter m has dimensions of length and equals GM/c^2 in ordinary units. The static exterior has $r > 2m$, with the usual separate angular coordinate charts.

Proposition 1. In the static orthonormal exterior frame, the tidal eigenvalues are $-2q, q, q$, where $q = m/r^3$. The spacetime is Ricci flat and

$$\mathcal{K} = 48q^2 = \frac{48m^2}{r^6}. \quad (8)$$

Proof. Use the orthonormal coframe $\vartheta^0 = \sqrt{f} dt$, $\vartheta^1 = dr/\sqrt{f}$, $\vartheta^2 = r d\theta$ and $\vartheta^3 = r \sin\theta d\phi$. With the convention (2), the independent nonzero curvature entries are

$$\begin{aligned} R_{0101} &= -2q, & R_{0202} &= R_{0303} = q, \\ R_{1212} &= R_{1313} = -q, & R_{2323} &= 2q. \end{aligned} \quad (9)$$

They follow from $f' = 2m/r^2$ and $f'' = -4m/r^3$: the time-radial component is $f''/2$, the time-angular components are $f'/(2r)$, the radial-angular components are $-f'/(2r)$ and the angular component is $(1-f)/r^2$.

Contracting with the orthonormal metric gives zero Ricci tensor. Each independent entry occurs four times in the full quadratic contraction. Each time index occurs twice in the relevant squared entries, so the signs from raising indices are positive. Thus the contraction is $4(4 + 1 + 1 + 1 + 1 + 4)q^2 = 48q^2$. Taking the time–spatial entries gives the asserted eigenvalues. This calculation is a standard property of the Schwarzschild solution (Carroll, 2004; Tong, 2019).

The static frame is tied to accelerated exterior observers. At an event, a freely falling observer can have the same instantaneous time like direction, but a static world line is not a geodesic. The relative-motion interpretation of (6) requires the freely falling congruence and its propagated frame. The distinction avoids interpreting an accelerated frame as a freely falling trajectory.

3.2 A differential horizon diagnostic

Since $\mathcal{K}$ depends only on r , (4) gives

$$\mathcal{D} = f(r) \left(\frac{d\mathcal{K}}{dr} \right)^2 = \frac{82944m^4}{r^{14}} \left(1 - \frac{2m}{r} \right). \quad (10)$$

Proposition 2. Within the positive-mass Schwarzschild family, at finite positive areal radius, $\mathcal{D}$ vanishes precisely at $r = 2m$. Furthermore, the pair $(\mathcal{K}, \mathcal{D})$ recovers both r and m through

$$q = \sqrt{\frac{\mathcal{K}}{48}}, \quad J = \frac{\mathcal{D}}{36\mathcal{K}^2}, \quad r = (J + 2q)^{-1/2}, \quad m = q(J + 2q)^{-3/2}. \quad (11)$$

Proof. Differentiate (8) to obtain $d\mathcal{K}/dr = -288m^2/r^7$. Squaring and multiplying by $g^{rr} = f$ proves (10). Its prefactor is nonzero for $m > 0$ and finite $r > 0$, so its only zero is the zero of f . Division by $36\mathcal{K}^2$ gives $J = f/r^2 = 1/r^2 - 2m/r^3$. Since $q = m/r^3$, it follows that $J + 2q = 1/r^2 > 0$. Solving for the positive radius and then the mass proves (11). The scalar expressions extend through the horizon in a regular chart.

These are family-dependent reconstruction formulae, not a proof that two invariants identify an arbitrary space-time. The assumption that the metric belongs to the positive-mass Schwarzschild family is essential, as is the exclusion of $m = 0$, where division by $\mathcal{K}$ would be invalid. Dimensional checks give $\mathcal{K}$ units of inverse length to the fourth power, $\mathcal{D}$ inverse length to the tenth power and both q and J inverse length squared. Formula (11) therefore returns the correct dimensions.

The use of curvature gradients is consistent with the stationary-horizon programme of Page and Shoom (2015). The calculation here is its elementary cohomogeneity-one illustration, not a new universal horizon detector. Other stationary families can have additional zero sets, and in a general dynamical space-time an event horizon is a global causal object.

3.3 Regularity at the horizon

At $r = 2m$, $\mathcal{K} = 3/(4m^4)$ is finite. To prove regularity of the metric rather than infer it from that scalar alone, define $v = t + r_*$, where $dr_*/dr = f^{-1}$. Then

$$ds^2 = -f(r)dv^2 + 2dvdr + r^2d\Omega^2. \quad (12)$$

The determinant of the time–radial block is -1 , including at $r = 2m$. The metric coefficients are smooth there in the ingoing extension. The scalar zero in (10) therefore marks a regular horizon in this extended solution, rather than a curvature singularity. At $r = 0$, (8) diverges. A smooth extension through a point at which this scalar diverges cannot preserve a regular smooth metric. This reasoning concerns this particular obstruction; it does not equate every type of geodesic incompleteness with scalar divergence.

4. HOMOGENEOUS COSMOLOGY AND A CLASS SPECIFIC FLATNESS CRITERION

Take the Friedmann–Lemaître–Robertson–Walker metric

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2d\Omega^2 \right], \quad a(t) > 0. \quad (13)$$

The radial coordinate is dimensionless, a has dimensions of length, and k is normalised to $-1, 0, 1$. Let $H = \dot{a}/a$ and define $X = \ddot{a}/a$ and $Y = H^2 + k/a^2$.

Proposition 3. In a comoving orthonormal frame,

$$R_{0i0j} = -X\delta_{ij}, \quad R_{ijkl} = Y(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \quad (14)$$

with no mixed time–spatial–spatial curvature entries. Consequently,

$$R = 6(X + Y), \quad \mathcal{K} = 12(X^2 + Y^2). \quad (15)$$

At an event, $\mathcal{K} = 0$ if and only if the full Riemann tensor vanishes there. On a region, vanishing $\mathcal{K}$ therefore gives local flatness throughout that region.

Proof. Homogeneity and isotropy leave one time–spatial coefficient and one purely spatial coefficient. Calculating the connection of (13) gives these coefficients as $-\ddot{a}/a$ and $(\dot{a}^2 + k)/a^2$, yielding (14). The three independent time–spatial sectional entries each have magnitude $|X|$, and the three spatial entries each have magnitude $|Y|$. Tensor symmetries supply the factor four, giving $\mathcal{K} = 12X^2 + 12Y^2$. Ricci contraction gives $R_{00} = -3X$ and $R_{ij} = (X + 2Y)\delta_{ij}$, whence the first identity in (15). Since X and Y are real, their squared sum vanishes exactly when both are zero; (14) then vanishes. The implication is restricted to this metric class.

Two examples clarify the distinction between the Ricci scalar and the complete curvature. For a spatially flat metric with $a(t) = a_0(t/t_0)^{1/2}$ on $t > 0$, one has $X = -1/(4t^2)$ and $Y = 1/(4t^2)$. Thus $R = 0$ but $\mathcal{K} = 3/(2t^4) > 0$. The vanishing Ricci scalar does not imply flatness, even in this highly symmetric class. For $k = -1$ and $a(t) = t$ on $t > 0$, both X and Y vanish; this is the locally flat Milne description, despite a nonconstant scale factor.

For a comoving perfect fluid, (3) reduces to $3Y = 8\pi G\rho + \Lambda$ and $X = -4\pi G(\rho + 3p)/3 + \Lambda/3$. These familiar equations (Carroll, 2004; Tong, 2019) link the geometric coefficients to a matter model. Proposition 3 does not require a fluid equation of state: it is a statement about the metric

geometry before imposing those dynamical relations. Keeping these logical steps separate distinguishes an invariant identity from a cosmological inference.

5. EXACT PLANE WAVES AND SCALAR BLINDNESS

5.1 A Ricci flat counterexample

Consider a Brinkmann metric with null coordinates u, v and transverse coordinates x, y :

$$ds^2 = -2 du dv + dx^2 + dy^2 + F(u, x, y) du^2. \quad (16)$$

The null block has determinant -1 , so the signature is Lorentzian for every real smooth F . Its inverse satisfies $g^{uu} = 0$, $g^{uv} = -1$ and $g^{vv} = -F$. Take the quadratic profile

$$F = A(u)(x^2 - y^2) + 2B(u)xy, \quad (17)$$

where A and B are smooth real functions.

Proposition 4. The metric (16)–(17) is Ricci flat. Every scalar polynomial contraction involving at least one factor of the Riemann tensor and no covariant derivatives vanishes. Nevertheless, the Riemann tensor is nonzero wherever $A^2 + B^2 > 0$.

Proof. The connection entries relevant to curvature are $\Gamma_{uu}^i = -\partial_i F/2$, $\Gamma_{ui}^v = -\partial_i F/2$ and $\Gamma_{uu}^v = -\partial_u F/2$, with i transverse. Direct substitution into (2) leaves, up to tensor symmetries,

$$R_{uiuj} = -\frac{1}{2} \partial_i \partial_j F, \quad R_{uu} = -\frac{1}{2} (\partial_x^2 F + \partial_y^2 F). \quad (18)$$

The transverse Laplacian of (17) vanishes, proving Ricci flatness. Its Hessian is twice the matrix with diagonal entries $A, -A$ and off-diagonal entries B , so curvature is nonzero when either function is nonzero.

Every nonzero covariant curvature entry has two u indices and no v index. In a complete contraction with inverse metrics, a lower u can pair only with a lower v , because $g^{uu} = g^{vi} = 0$. No curvature factor supplies such a v index. Thus every complete algebraic contraction of a positive number of curvature factors vanishes. In particular, $R = R_{\mu\nu} R^{\mu\nu} = \mathcal{K} = 0$. This is an explicit example of scalar blindness, consistent with the exceptional Lorentzian behaviour studied in the work of Coley, Hervik and Pelavas (2009).

The proposition establishes only algebraic contractions. No claim about invariants containing arbitrary covariant derivatives is needed for the counterexample, and the contraction proof should not be treated as a proof of that broader statement.

5.2 Nonzero tidal information

Along $x = y = 0$, take $u = v = \tau/\sqrt{2}$ and constant transverse coordinates. Here F and its first derivatives vanish, so this curve is a unit timelike geodesic and $U = (\partial_u + \partial_v)/\sqrt{2}$. The transverse basis is orthonormal and parallel propagated along it. Equations (5) and (18) give

$$E_{ij} = -\frac{1}{2} \begin{pmatrix} A & B \\ B & -A \end{pmatrix}_{ij}, \quad i, j \in \{x, y\}. \quad (19)$$

Its transverse eigenvalues are $\pm\sqrt{A^2 + B^2}/2$, and the longitudinal eigenvalue is zero. Thus the relative acceleration of transverse separations is nonzero wherever the wave profile is nonzero, even though the algebraic scalar invariants in Proposition 4 vanish.

This exact example complements the familiar linearized-wave discussion (Flanagan and Hughes, 2005). It does not assume a small wave amplitude. The geodesic-deviation equation still concerns infinitesimal neighboring separations, so it is not an arbitrary finite-detector response formula. The eigenvalues belong to the stated observer; changing the time like direction can change the measured tidal matrix. Rotating that observer's spatial frame does not change its eigenvalues.

6. COMPARATIVE INTERPRETATION AND CURRENT MATHEMATICAL CONTEXT

6.1 Conditional diagnostics

Metric class	Scalar result	Additional information	Valid conclusion
Positive-mass Schwarzschild	Kretschmann scalar is positive; gradient norm changes sign at the horizon	Family assumption and a regular horizon extension	Recover radius and mass; identify the horizon within this family
Homogeneous isotropic metric	Kretschmann scalar is a sum of two real squares	Symmetry restrictions on all curvature components	Zero scalar implies zero curvature within this class
Exact vacuum plane wave	Algebraic curvature contractions vanish	Nonzero Hessian and observer tidal matrix	Vanishing scalars do not imply general Lorentzian flatness

Table 1. Comparison derived from Propositions 1–4. These are mathematical identities, not empirical measurements. Propositions 3 and 4 are not contradictory. The former depends on isotropy, which gives a restricted component structure and a positive sum of squares. The latter exploits null index alignment, which prevents nonzero curvature components from surviving a complete contraction. Lorentzian signature permits this mechanism; scalar invariance under coordinates does not guarantee completeness as a diagnostic. Similarly, Proposition 2 does not contradict the global definition of an event horizon. The global Schwarzschild extension and its causal interpretation are already part of the solution under consideration. The local invariant recognizes a hypersurface in that known family. It does not discover an event horizon in an otherwise unspecified space-time. This is the distinction between a conditional geometric diagnostic and a universal causal definition.

6.2 Relation to invariant characterization

The computations illustrate why the invariant-characterization question in the work of Coley, Hervik and Pelavas (2009) requires more than computing a small number of contractions. One must distinguish the full set of scalar polynomial invariants from a chosen diagnostic subset, and distinguish local metric characterization from global topology. Invariant reconstruction in (11) is deliberately restricted to two parameters at an event in an already specified family. It does not imply that those two quantities establish membership in that family. The plane-wave example also motivates retaining curvature components in geometrically specified frames when scalars lose information. Observer-relative tidal tensors provide one physically interpretable alternative. A complete local equivalence analysis requires additional frame-fixing and derivative information, which is outside the elementary proofs given here. The comparison therefore supports careful diagnostic selection rather than replacement of the established classification programme.

6.3 Regularity and singularity conclusions

The smooth assumptions used above ensure that classical curvature components and coordinate changes are well defined. Recent work extends incompleteness theorems to weaker regularity (Kunzinger, Ohanyan, Schinnerl and Steinbauer, 2022). At continuously differentiable metric regularity, distributional curvature and loss of uniqueness for geodesics require additional treatment. The same smooth component formulae cannot simply be inserted into that setting without defining their weak meaning. Steinbauer's survey (2023) explains how focusing, causal assumptions and regularity enter singularity theorems. Penrose's collapse theorem (1965) provides the foundational example. The appropriate conclusion is geodesic incompleteness under specified hypotheses, rather than the unconditional divergence of a selected scalar. Our Schwarzschild center exhibits scalar divergence, but that example is not a definition of every singular behaviour. These contemporary developments sharpen the limits of the diagnostic comparison. Scalar values, differentiability assumptions and global causal hypotheses address different aspects of a spacetime. A rigorous application must state which of them supports each conclusion.

7. SCOPE OF THE ANALYTICAL RESULTS

This survey gives explicit proofs for selected smooth metrics rather than a new classification theorem. It does not cover rotating black holes, generic nonstationary horizons, complete Cartan equivalence algorithms or arbitrary weak-curvature spaces. The plane-wave proof concerns algebraic contractions; a classification of all derivative invariants is not derived. No observation or numerical experiment tests the diagnostic formulae. The reconstruction result is conditional on a specified metric family, the cosmological flatness criterion is conditional on homogeneity and isotropy, and the plane-wave counterexample applies to algebraic curvature contractions. These boundaries are part of the conclusions themselves. The comparison establishes the information supplied by each diagnostic without extending that information to an unrestricted metric-equivalence test or a global singularity criterion.

8. CONCLUSION

Differential geometry connects general relativity to curvature diagnostics whose information content depends on assumptions. Schwarzschild curvature and its gradient permit explicit reconstruction within a known positive-mass family. Homogeneous isotropy produces a restricted flatness criterion. Exact vacuum plane waves show that vanishing algebraic scalar curvature invariants cannot imply flatness for arbitrary Lorentzian metrics, while a specified observer still measures nonzero tidal effects. Local diagnostics must therefore be distinguished from global horizon definitions and incompleteness theorems. The comparative proofs make those dependencies explicit and provide a focused mathematical account of the limits and uses of curvature in general relativity.

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